4.1 Derivability at a Point

A real valued function f defined on an interval $I \subseteq \mathbb{R}$ is said to be differentiable at a point $c \in I$ if for any $\epsilon > 0$, $\exists$ $\delta(\epsilon) > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon, \text{ when } 0 < |x - c| < \delta$$

where $x \in I$

In other words, a real valued function f defined on an interval I is said to differentiable at a point $c \in I$, if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists (finitely)}$$

This limit, in case it exists, is denoted by $f'(c)$ and

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is called derivative of f at c

Left Hand Derivative: The left hand derivative of f at $x=c$ is defined as

$$\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}, \quad h>0$$

and is denoted by $f'(c-0)$ or $Lf'(c)$

Right Hand Derivative. The right hand derivative of f at $x=c$ is defined as

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}, \quad h>0$$

and is denoted by $f'(c+0)$ or $Rf'(c)$

Thus derivative $f'(c)$ exists iff $Lf'(c) = Rf'(c)$

4.2 Differentiability in an interval

(1) A function $f:[a,b]\longrightarrow R$ is said to be ~~derivable~~ differentiable at the end points a and b if $Rf'(a)$ and $Lf'(b)$ exist respectively i.e.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h\to 0} \frac{f(b-h)-f(b)}{-h} \text{ exist}$$

respectively.

(2) A function $f:[a,b]\longrightarrow R$ is said to be derivable in an interval $[a,b]$ if it is

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derivable at the end points of a and b and it is derivable at the each point of (a,b).

Theorem 4.1 If a function $f: I \to R$ is differentiable at $c \in I$. Then f is continuous at c but converse need not be true.

Proof: Since f is differentiable at $c \in I$

$\Rightarrow$ $f'(c)$ exists

$$\Rightarrow \quad f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad -(1)$$

Now

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \times (x - c) \quad , x \neq c$$

Taking limits as $x \to c$ on both sides

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} . (x - c) \right\}$$

$$= \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] . \lim_{x \to c} (x - c)$$

$$= f'(c) . 0 \quad , \text{ from (1)}$$

$$= 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

Hence f is continuous at c.

Consider the function f defined by

$$f(x) = |x| , \forall x \in R$$

$f(x)$ can be written as

$$f(x) = \begin{cases} -x & , \text{if } x < 0 \\ 0 & , \text{if } x = 0 \\ x & , \text{if } x > 0 \end{cases}$$

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At $x=0$,

$$LHL = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -(0-h) = 0$$

$$RHL = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h) = 0$$

$$\Rightarrow f(0-0) = f(0+0) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 \quad \text{---(II)}$$

But $f(0) = 0$ ---(III)

From (II), III, we get

$$\lim_{x\to 0} f(x) = f(0)$$

Hence f is continuous at $x=0$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-(0-h)-0}{-h}$$

$$= \lim_{h\to 0} (-1) = -1$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)-0}{h}$$

$$= \lim_{h\to 0} (1) = 1$$

$$\Rightarrow Lf'(0) \neq Rf'(0)$$

Therefore f is not differentiable at $x=0$

Hence every continuous function need not be differentiable ~~at~~.

4.3 Algebra of Differentiable function

Theorem 4.2 Let The function $f: I \to R$ and $g: I \to R$, where I be an interval be ~~Then~~ diff. at $c \in I$. Then

(a) If $\alpha \in R$, then the function αf is differentiable at

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$c \in I$ and $(\alpha f)'(c) = \alpha f'(c)$

(b) The function $f \pm g$ is differentiable at c and

$$(f \pm g)'(c) = f'(c) \pm g'(c)$$

(c) (Product Rule) The function fg is diff. at $c \in I$ and $(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$

(d) (Quotient Rule) If $g(c) \neq 0$, then function $\frac{f}{g}$ is differentiable at $c \in I$. and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c) f'(c) - f(c) g'(c)}{[g(c)]^2}$$

Proof. (a) Given that f is diff. at c

$\Rightarrow f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists

If $\alpha \in R$, then

$$\lim_{x \to c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c} = \lim_{x \to c} \frac{\alpha f(x) - \alpha f(c)}{(x - c)}$$

$$\Rightarrow \quad (\alpha f)'(c) = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \alpha f'(c)$$

$\Rightarrow (\alpha f)'(c)$ exists as $f'(c)$ exists

$\Rightarrow \alpha f$ is diff. at c and $(\alpha f)'(c) = \alpha f'(c)$

(b) Given that f and g are diff. at c

$\Rightarrow f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ and $g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$

both are exist

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Now

$$\lim_{x\to c}\frac{(f+g)(x)-(f+g)(c)}{x-c}=\lim_{x\to c}\frac{f(x)-f(c)+g(x)-g(c)}{x-c}$$

$$\Rightarrow\quad (f+g)'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}+\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$\Rightarrow\quad (f+g)'(c) = f'(c)+g'(c)$$

$\Rightarrow$ $f+g$ is diff at c and $(f+g)'(c)=f'(c)+g'(c)$

Similarly we can prove that $f-g$ is diff at c and $(f-g)'(c)=f'(c)-g'(c)$.

(c). Since f & g are diff at c, therefore

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ and } g'(c)=\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

Now

$$\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{(x-c)}$$

$$=\lim_{x\to c}\left[f(x)\cdot\frac{g(x)-g(c)}{x-c}+\frac{f(x)-f(c)}{x-c}\cdot g(c)\right]$$

$$=f(c)\,g'(c)+f'(c)\,g(c)$$

$\Rightarrow$ $\displaystyle\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}$ exists

$\Rightarrow$ fg is diff at c and

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$(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

(d) Let $h = \frac{f}{g}$, then

$$h(x) - h(c) = \left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c) = \frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}$$

$$= \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}$$

$$= \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)}$$

$$= \frac{f(x) - f(c)}{g(x)g(c)} \cdot g(c) - f(c) \cdot \frac{g(x) - g(c)}{g(x)g(c)}$$

Now

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot \frac{1}{g(x)} - \frac{f(c)}{g(c)} \frac{g(x) - g(c)}{g(x)(x-c)}\right]$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \frac{1}{g(c)} - \frac{f(c)}{[g(c)]^2} \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= f'(c) \cdot \frac{1}{g(c)} - \frac{f(c)}{[g(c)]^2} \cdot g'(c)$$

[$\because$ f & g are diff at c]

$$= \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

$\Rightarrow \lim_{x \to c} \frac{h(x) - h(c)}{x - c}$ exists

$\Rightarrow h = \frac{f}{g}$ is diff. at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

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